ON THE DIOPHANTINE EQUATION $\alpha^l + \beta^l = C\gamma^{l^n}$

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In this paper we designate by

l	an odd prime number,
ζ_ν	a primitive l^ν-*th* root of unity, $\zeta_\nu = e^{2\pi i/l^\nu}$
k	the cyclotomic field of ζ_1,
k_0	the real subfield of k of the degree $(l-1)/2$,
$h = h_1 l^t$	the class number of k, $l \nmid h_1$,
h_0	the class number of k_0,
$K = k(\zeta_{t+1})$	the cyclotomic field of ζ_{t+1},
$\lambda_{t+1} = 1 - \zeta_{t+1}$	the prime divisor of l in K,
${}_K\left(\frac{\omega}{\mathfrak{p}}\right)_{(\nu)}$	the l^ν-*th* power charactor in K,
$\bar{\alpha}$	the conjugate complex number to α.

THEOREM. *For* $n \geqq 3t+2$, $2m > t+3$ *and* $(\varphi\{(C)\}, l) = 1$,

(1) $$\alpha^l + \beta^l = C\varepsilon(\lambda\bar{\lambda})^{ml}\gamma^{l^n}$$

is impossible in non-zero integer α, β *and* γ *in* k_0, *where* ε *is a real unit and* $\lambda = 1 - \zeta_1$, *and* $\varphi\{(C)\}$ *denotes Euler's function of principal ideal* (C) *in* k *and* $(\alpha, \beta) = 1$, $l \nmid h_0$.

REMARK. For $C = 1$, this theorem was proved by Prof. T. Morishima [1].

Proof. I. Yamaguchi [2] proved the following lemma:

LEMMA. *Let* $(\varphi\{(C)\}, l) = 1$ *and assume that the equation* (1) *is solvable* *in integer* α, β *and* γ *in* $Q(\zeta+\zeta^{-1})$ *which is the maximal real subfield of* l*-th cyclotomic field* $Q(\zeta)$. *Then the factor* $\alpha+\beta$ *of the left side of* (1) *is divisible by* (C). *Where* $\varphi\{(C)\}$ *is the Euler function of principal ideal* (C) *in* $Q(\zeta)$.

By this lemma, $\alpha+\beta$ is divisible by C. Since $h=h_1l^t$ $(l\nmid h_1)$, $l\nmid h_0$ and $\alpha+\beta$ is an integer in k_0, we have

(2) $$\frac{\alpha+\zeta_1{}^r\beta}{1-\zeta_1{}^r}=\varepsilon_r\omega_r{}^{l^{n-t}}$$

(3) $$\frac{\alpha+\zeta_1{}^{-r}\beta}{1-\zeta_1{}^{-r}}=\bar{\varepsilon}_r\bar{\omega}_r{}^{l^{n-t}}$$

(4) $$\alpha+\beta=C\varepsilon_0\lambda^{(2m-1)l+1}\omega_0{}^{l^n},$$

where ε's are real units.

Hence from (2), (3) and (4)

(5) $$\omega_r{}^{l^{n-t}}-\bar{\omega}_r{}^{l^{n-t}}=C\varepsilon'\lambda^{(2m-1)l}\omega_0{}^{l^n},$$

therefore

$$\prod_{i=0}^{l^{t+1}-1}(\omega_r^{l^{n-2t-1}}-\zeta_{t+1}^i\bar{\omega}_r^{l^{n-2t-1}})=C\varepsilon'\lambda_{l^{t+1}}^{(2m-1)l^{t+1}}\omega_0{}^{l^n}.$$

Consider that

$$\left(\frac{\omega_r{}^{l^{n-2t-1}}-\zeta^i{}_{t+1}\bar{\omega}^{l^{n-2t-1}}}{1-\zeta^i{}_{t+1}}\right)=(\Omega)=\mathfrak{A}_i{}^{l^n},$$

then since

$$\Omega\equiv\omega_r^{l^{n-2t-1}}\pmod{\lambda_{t+1}^{(2m-2)l^{t+1}+1}}$$

$n-2t-1\geqq t+1$, $(2m-2)l^{t+1}+1>(t+1)(l-1)l^t+l^t+3$, for all prime ideal $\mathfrak{p}$ prime to $l\Omega$ in k [3],

$${}_K\left(\frac{\Omega}{\mathfrak{p}^h}\right)_{(t+1)}={}_K\left(\frac{\Omega}{\theta}\right)_{(t+1)}={}_K\left(\frac{\theta}{\Omega}\right)_{(t+1)}={}_K\left(\frac{\theta}{\mathfrak{A}_i}\right)_{(t+1)}^{l^n}=1,$$

where $\mathfrak{p}^h=(\theta)$.

Since

$$1={}_K\left(\frac{\Omega}{\mathfrak{p}}\right)_{(t+1)}^{h_1 l^t}={}_K\left(\frac{\Omega}{\mathfrak{p}}\right)_{(1)}^{h_1}={}_k\left(\frac{N_{K/k}(\Omega)}{\mathfrak{p}}\right)_{(1)}^{h_1},$$

then by [4],

$$N_{K/k}(\Omega)=\frac{\omega_r{}^{l^{n-t-1}}-\zeta_1{}^i\bar{\omega}_r{}^{l^{n-t-1}}}{1-\zeta_1{}^i}=\delta_i{}^l,$$

and from (5)

$$\omega_r{}^{l^{n-t-1}}-\bar{\omega}_r{}^{l^{n-t-1}}=C\varepsilon''\lambda^{(2m-2)l+1}\delta_0{}^{l^n},$$

for $\qquad \zeta_1{}^{-(l-1)/2}\lambda^{-1}(\omega_r{}^{l^{n-t-1}}-\bar{\omega}_r{}^{l^{n-t-1}})\in k_0,$

then

(6) $\qquad \omega_r{}^{2l^{n-t-1}}+\bar{\omega}_r{}^{2l^{n-t-1}}-(\zeta_1{}^i+\zeta_1{}^{-i})\,\omega_r{}^{l^{n-t-1}}\bar{\omega}_r{}^{l^{n-t-1}}$

$$=(1-\zeta_1{}^i)(1-\zeta_1{}^{-i})(\delta_i\delta_{-i})^l,$$

(7) $\qquad \omega_r{}^{2l^{n-t-1}}+\bar{\omega}_r{}^{2l^{n-t-1}}-2\omega_r{}^{l^{n-t-1}}\bar{\omega}_r{}^{l^{n-t-1}}$

$$=C^2\varepsilon'''\lambda^{2(2m-2)l+2}\delta_0{}^{2l^n}.$$

From (6), (7) for $i=1,2$,

$$(\delta_1\delta_{-1})^l-(\delta_2\delta_{-2})^l$$

$$=\frac{(\zeta_1+\zeta_1{}^{-1}-\zeta_1{}^2-\zeta_1{}^{-2})(\omega_r{}^{2l^{n-t-1}}+\bar{\omega}_r{}^{2l^{n-t-1}}-2\omega_r{}^{l^{n-t-1}}\bar{\omega}_r{}^{l^{n-t-1}})}{(1-\zeta_1)(1-\zeta_1{}^{-1})(1-\zeta_1{}^2)(1-\zeta_1{}^{-2})}$$

$$=C^2\varepsilon''''\lambda^{2(2m-2)l}\delta_0{}^{2l^n},$$

then

$$\alpha_1{}^l+\beta_1{}^l=C^2\varepsilon_1(\lambda\bar{\lambda})^{(2m-2)l}\gamma_1{}^{l^n},$$

where $\alpha_1,\ \beta_1,\ \gamma_1\in k_0,\ (\alpha_1,\ \beta_1)=1,\ 2(2m-2)>t+3,$ which is a contradiction.

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