ON FERMAT'S LAST THEOREM

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1. Introduction

Wieferich[7] and Mirimanoff[2] proved the following Lemma:

LEMMA 1. *If the equation*

(1) $$x^l+y^l+z^l=0$$

has an integral solution with $(xyz, l)=1$, *then*

(2) $$2^{l-1}\equiv 1 \pmod{l^2}$$

and

(3) $$3^{l-1}\equiv 1 \pmod{l^2},$$

where (a, b) stands for the greatest common divisor a and b.

Also Perisastri[4] proved following lemma:

LEMMA 2. *If* $t>1$ *an integer and* α *any integer, then there exists a polynomial g(x) with integral coefficients such that*

(4) $$x^t=(x-\alpha)^2\, g(x)+t\alpha^{t-1}x+\alpha^t(1-t).$$

Using Lemma 1 and Lemma 2, we shall prove the following Theorem:

THEOREM 1. *Let n be a prime. If* $l=3^n-2$ *is a prime, then the equation (1) has no integral solutions with* $(xyz, l)=1$.

In Theorem 1, $l-1\equiv 0 \pmod n$; for, if $n=2$, then $l-1=3^2-2-1=6\equiv 0 \pmod 2$ and if $n>2$, then by Fermat's theorem, $3^n\equiv 3 \pmod n$ and therefore $l-1=3^n-2-1=3^n-3\equiv 0 \pmod n$.

2. Proof of Theorem 1

Suppose that the equation (1) has an integral solution with $(xyz,l)=1$. Without loss of generality we can suppose that x, y, z are pairwise prime. Then putting $t=l-1$, $\alpha=2$ and $x=3^n$ in (4), it follows that

$$(3^n)^{l-1}=(3^n-2)^2g(3^n) + (l-1)2^{(l-1)-1}3^n+2^{l-1}\left\{1-(l-1)\right\},$$

$$3^{n(l-1)}-1=l^2g(3^n)+l(l-1)2^{l-2}+(2^{l-1}-1). \tag{5}$$

Hence Lemma 1 and (5) imply $l(l-1)2^{l-2}\equiv 0 \pmod{l^2}$, which contradicts that $(l-1)2^{l-2}\not\equiv 0 \pmod l$. Thus Theorem is proved.

3. The extension of Theorem 1

Wieferich[7], Mirimanoff[2], Vandiver[6], Frobenius[1], Pollaczek [5] and Morishima[3] proved the following Lemma:

LEMMA 3. *If the equation (1) has an integral solution with $(xyz, l)=1$, then $m^{l-1}\equiv 1 \pmod{l^2}$ for all primes $2\leqq m\leqq 43$.*

Using Lemma 1 and Lemma 2, we can prove the following Theorem:

THEOREM 2. *Let n be a prime. If $l=p^n-(p-1)$ is a prime, then the equation (1) has no integral solutions with $(xyz, l)=1$, where p is a prime and $2\leqq p\leqq 43$.*

REFERENCES

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